

RANDOM ORTHONORMAL BASES OF SPACES OF HIGH DIMENSION

STEVE ZELDITCH

ABSTRACT. We consider a sequence \mathcal{H}_N of finite dimensional Hilbert spaces of dimensions $d_N \rightarrow \infty$. Motivating examples are eigenspaces, or spaces of quasi-modes, for a Laplace or Schrödinger operator on a compact Riemannian manifold. The set of Hermitian orthonormal bases of \mathcal{H}_N may be identified with $U(d_N)$, and a random orthonormal basis of $\bigoplus_N \mathcal{H}_N$ is a choice of a random sequence $U_N \in U(d_N)$ from the product of normalized Haar measures. We prove that if $d_N \rightarrow \infty$ and if $\frac{1}{d_N} \text{Tr} A|_{\mathcal{H}_N}$ tends to a unique limit state $\omega(A)$, then almost surely an orthonormal basis is quantum ergodic with limit state $\omega(A)$. This generalizes an earlier result of the author in the case where \mathcal{H}_N is the space of spherical harmonics on S^2 . In particular, it holds on the flat torus $\mathbb{R}^d/\mathbb{Z}^d$ if $d \geq 5$ and shows that a highly localized orthonormal basis can be synthesized from quantum ergodic ones and vice-versa in relatively small dimensions.

The purpose of this article is to prove a general result on the quantum ergodicity of random orthonormal bases $\{\psi_{N,j}\}_{j=1}^{d_N}$ of finite dimensional Hilbert spaces $\mathcal{H}_N \subset L^2(M)$ of dimensions $d_N \rightarrow \infty$ of a compact Riemannian manifold (M, g) . The proof is based on a “moment polytope” interpretation of quantum ergodicity from [Z1]: the quantum variances of a Hermitian observable $A \in \Psi^0(M)$ are identified with moments of inertia of the convex polytopes $\mathcal{P}_{\vec{\lambda}}$ defined as the convex hull of the vectors $\vec{\lambda} = (\lambda_1, \dots, \lambda_{d_N})$ of eigenvalues (in all possible orders) of $\Pi_N A \Pi_N$ where $\Pi_N : L^2(M) \rightarrow \mathcal{H}_N$ is the orthogonal projection. Equivalently, $\mathcal{P}_{\vec{\lambda}}$ is the image of the coadjoint orbit $\mathcal{O}_{\vec{\lambda}}$ of the diagonal matrix $D(\vec{\lambda})$ under the moment map for the Hamiltonian action of the maximal torus $T_{d_N} \subset U(d_N)$ of diagonal matrices acting by conjugation on $\mathcal{O}_{\vec{\lambda}}$. In particular, the main estimates of quantum ergodicity can be formulated in terms of estimates of the first four moments of inertia of $\mathcal{P}_{\vec{\lambda}}$. The main result, Theorem 1, states that random orthonormal bases are almost surely quantum ergodic as long as $d_N \rightarrow \infty$ and $\frac{1}{d_N} \text{Tr} \Pi_N A \Pi_N \rightarrow \omega(A)$ for all $A \in \Psi^0(M)$, where $\omega(A)$ is the Liouville state. More generally, if these traces have any unique limit state, then almost surely it is the quantum limit of a random orthonormal basis. The proof is essentially implicit in [Z1], but we bring it out explicitly here and also give detailed calculations of the moments of inertia, which seem of independent interest.

Quantum ergodicity of random orthonormal bases is a rigorous result on the ‘random wave model’ in quantum chaos, according to which eigenfunctions of quantum chaotic systems should behave like random waves. It also has implications for the approximation of modes by quasi-modes. Since eigenfunctions of the Laplacian Δ of a compact Riemannian manifold (M, g) form an orthonormal basis, it is natural to compare the orthonormal basis of eigenfunctions to a ‘random orthonormal basis’. In [Z1], the result of this article was proved for the special case where \mathcal{H}_N is the space of degree N spherical harmonics on the standard

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S^2 . In [Z2] the quantum ergodic property was generalized to any compact Riemannian manifold, with \mathcal{H}_N the span of the eigenfunctions in a spectral interval $[N, N+1]$ for $\sqrt{\Delta}$. Related results have recently been proved in [SZ, BL]. The dimension of such \mathcal{H}_N grows at the rate N^{m-1} where $m = \dim M$ and thus a random element of \mathcal{H}_N is a superposition of N^{m-1} states. The results of this article show that the same quantum ergodicity property holds for sequences of eigenspaces (or linear combinations) whose dimensions d_N tend to infinity at any rate. For instance, the results show that random orthonormal bases of eigenfunctions on a flat torus of dimension ≥ 5 are quantum ergodic (for the precise statement, see §4.1, and for further discussion, see §0.1.)

To explain the moment map interpretation and the variance formula, recall that quantum ergodicity is concerned with quantum variances, i.e. with the dispersion from the mean of the diagonal part of a Hermitian matrix H_N on a large dimensional vector space \mathcal{H}_N . The matrix H_N is the restriction

$$T_N^A := \Pi_N A \Pi_N \quad (0.1)$$

to \mathcal{H}_N of a pseudo-differential operator $A \in \Psi^0(M)$; here Π_N is the orthogonal projection to \mathcal{H}_N and $\Psi^0(M)$ is the space of pseudo-differential operators of order zero. The same methods and results apply to other context such as semi-classical pseudo-differential operators or to Toeplitz operators on holomorphic sections of powers of a positive line bundle [SZ]. Given an ONB $\{\psi_{N,j}\}_{j=1}^{d_N}$ of \mathcal{H}_N we define the quantum variances of the ONB (indexed by $A \in \Psi^0(M)$) by

$$V_A(\{\psi_{N,k}\}) := \frac{1}{d_N} \sum_{j=1}^{d_N} |\langle A \psi_{N,j}, \psi_{N,j} \rangle - \omega(A)|^2. \quad (0.2)$$

Here, $\omega(A) = \int_{S^*M} \sigma_A d\mu_L$ where $d\mu_L$ is normalized Liouville measure (of mass one).

Definition: A sequence $\{\psi_{N,j}\}_N$ of ONB's of \mathcal{H}_N is a quantum ergodic ONB of $L^2(M)$ if

$$(\mathcal{EP}) \quad \lim_{N \rightarrow \infty} V_A(\{\psi_{N,k}\}) = 0, \quad \forall A \in \Psi^0(M). \quad (0.3)$$

By a standard diagonal argument, this implies that almost all the individual elements $\langle A \psi_{N,j}, \psi_{N,j} \rangle$ tend to $\omega(A)$. Since this aspect of quantum ergodicity is the same as in [Z1, SZ] (e.g.) we do not discuss it here.

To define random orthonormal bases, we introduce the probability space $(\mathcal{ONB}, d\nu)$, where \mathcal{ONB} is the infinite product of the sets \mathcal{ONB}_N of orthonormal bases of the spaces \mathcal{H}_N , and $\nu = \prod_{N=1}^{\infty} \nu_N$, where ν_N is Haar probability measure on \mathcal{ONB}_N . A point of \mathcal{ONB} is thus a sequence $\Psi = \{(\psi_1^N, \dots, \psi_{d_N}^N)\}_{N \geq 1}$ of orthonormal basis. Given one orthonormal basis $\{e_j^N\}$ of \mathcal{H}_N any other is related to it by a unique unitary matrix. So the probability space is equivalent to the product

$$(\mathcal{ONB}, d\nu) \simeq \prod_{N=1}^{\infty} (U(d_N), dU) \quad (0.4)$$

where dU is the unit mass Haar measure on $U(d_N)$. Here we are working with Hermitian orthonormal bases and Hermitian pseudo-differential operators. We could also work with real self-adjoint operators and real orthonormal bases, which are then related by the orthogonal group. The results in that setting are essentially the same but the proofs are somewhat more complicated; for expository simplicity we stick to the unitary Hermitian framework.

Let $A \in \Psi^0$ and denote the eigenvalues of T_N^A by $\lambda_1, \dots, \lambda_{d_N}$. The empirical measure of eigenvalues of T_N^A is defined by

$$\nu_{\vec{\lambda}_N} := \frac{1}{d_N} \sum_{j=1}^{d_N} \delta_{\lambda_j}. \quad (0.5)$$

Its moments are given by

$$p_k(\lambda_1, \dots, \lambda_{d_N}) = \sum_{j=1}^{d_N} \lambda_j^k = \text{Tr}(T_N^A)^k. \quad (0.6)$$

To obtain quantum ergodicity, we put the following constraint on the sequence $\{\mathcal{H}_N\}$:

Definition: We say that \mathcal{H}_N has local Weyl asymptotics if, for all $A \in \Psi^0(M)$,

$$\frac{1}{d_N} \text{Tr} T_N^A = \omega(A) + o(1). \quad (0.7)$$

In fact, the results generalize to the case where $\omega(A)$ is replaced by any other limit state, i.e. $\int_{S^*M} \sigma_A d\mu$ where $d\mu$ is another invariant probability measure for the geodesic flow.

Our main result is:

THEOREM 1. *Let \mathcal{H}_N be a sequence of subspaces of $L^2(M)$ of dimensions $d_N = \dim \mathcal{H}_N \rightarrow \infty$. Assume that $\frac{1}{d_N} \text{Tr} \Pi_N A \Pi_N = \omega(A) + o(1)$ for all $A \in \Psi^0(M)$. Then with probability one in $(\mathcal{ONB}, d\nu)$, a random orthonormal basis of $\bigoplus_N \mathcal{H}_N$ is quantum ergodic.*

A natural question (which we do not study here) is whether a random orthonormal basis is QUE, i.e. whether

$$\max\{|\langle A\psi_{N,j}, \psi_{N,j} \rangle - \omega(A)|^2, \quad j = 1, \dots, d_N\} \rightarrow 0 \quad (a.s.) d\nu \quad ?.$$

As a tail event the probability of a random orthonormal basis being QUE is either 0 or 1.

We now explain how to formula Theorem 1 in terms of moment maps and polytopes. Quantum ergodicity of a random orthonormal bases concerns the dispersion from the mean of the diagonal part of T_N^A . The diagonal part depends on the choice of an orthonormal basis of \mathcal{H}_N . Once an orthonormal basis is fixed, iT_N^A can be identified with an element H_N of the Lie algebra $\mathfrak{u}(d_N)$ of $U(d_N)$, and a unitary change of the orthonormal basis results in the conjugation $H_N \rightarrow U_N^* H_N U_N$ of H_N . If the vector of eigenvalues of H_N is denoted $\vec{\lambda}_N$, then the conjugates sweep out the orbit $\mathcal{O}_{\vec{\lambda}_N}$. Let $\mathfrak{t}(d_N)$ denote the Cartan subalgebra of diagonal elements in $\mathfrak{u}(d_N)$, and let $\|\cdot\|^2$ denote the Euclidean inner product on $\mathfrak{t}(d_N)$. Also let

$$J_{d_N} : i\mathfrak{u}(d_N) \rightarrow i\mathfrak{t}(d_N)$$

denote the orthogonal projection (extracting the diagonal). Extracting the diagonal from each element of the orbit is precisely the moment map

$$J_{d_N} : \mathcal{O}_{\vec{\lambda}_N} \rightarrow \mathcal{P}_{\vec{\lambda}_N} \subset i\mathfrak{t}(d_N), \quad J_{d_N}(UD(\vec{\lambda})U^*) = \left(\dots, \sum_{j=1}^{d_N} \lambda_j |U_{ij}|^2, \dots \right) \quad (0.8)$$

of the conjugation action of T_{d_N} . Finally, let

$$\bar{J}_{d_N}(H) = \left(\frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N}, \quad D_0(\vec{\lambda}_N) = D(\vec{\lambda}_N) - \left(\frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N},$$

for Hermitian matrices $H \in i\mathfrak{u}(d_N)$. We also introduce notation for the diagonal of $D_0(\vec{\lambda})$:

$$D_0(\vec{\lambda}) = D(\vec{\Lambda}), \quad \text{with; } \Lambda_j := \lambda_j - \frac{1}{d_N} \sum_{j=1}^{d_N} \lambda_j. \quad (0.9)$$

Thus,

$$H = H^0 + \bar{J}_d(H), \quad \text{resp.} \quad D(\vec{\lambda}_N) = D_0(\vec{\lambda}_N) + \left(\frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N}$$

with H^0 traceless, corresponds to the decomposition $\mathfrak{u}(d_N) = \mathfrak{su}(d_N) \oplus \mathbb{R}$.

As this description indicates, quantum ergodicity of random orthonormal bases is mainly a result about the asymptotic geometry of the polytopes $\mathcal{P}_{\vec{\lambda}_N}$ corresponding to a sequence T_N^A of Toeplitz operators. The pushforward of the $U(d_N)$ -invariant normalized measure on $\mathcal{O}_{\vec{\lambda}}$ to $\mathcal{P}_{\vec{\lambda}}$ is the so-called Duistermaat-Heckman measure $d\mathcal{L}_{\vec{\lambda}}^{DH}$, a piecewise polynomial measure on $\mathcal{P}_{\vec{\lambda}}$. To prove almost sure quantum ergodicity, we prove that for all such sequences T_N^A and their spectra $\{\vec{\lambda}_N\}$, the second and fourth moments of inertia of $\mathcal{P}_{\vec{\lambda}}$ with respect to $d\mathcal{L}_{\vec{\lambda}}^{DH}$ are bounded. We use the property in Definition 0.7 to replace $\omega(A)$ by the centers of mass, i.e. the scalar matrix with the same trace as T_N^A . The Kolmogorov strong law of large numbers then gives the quantum ergodicity property. In [Z3], we study higher moments and their implication for the limit shape of $\mathcal{P}_{\vec{\lambda}}$ along a sequence $\{\lambda_N\}$ with a limit empirical measure.

We asymptotically evaluate the moments using the Fourier transform

$$\hat{\mu}_{\vec{\lambda}}(X) := \int_{\mathcal{O}(\vec{\lambda})} e^{i\langle X, \text{diag}(Y) \rangle} d\mu_{\vec{\lambda}}(Y) \quad (0.10)$$

of the δ -function on $\mathcal{O}_{\vec{\lambda}}$. Here, we assume $X \in \mathbb{R}^{d_N}$. We may identify X with a diagonal matrix, and then $\langle X, \text{diag}(Y) \rangle = \text{Tr } XY$, and we get the standard Fourier transform. We obviously have:

LEMMA 1. *Let Δ be the Euclidean Laplacian of \mathbb{R}^{d_N} acting in the X variable. Then,*

$$\begin{cases} m_2(\mathcal{P}_{\vec{\lambda}_N}) := \mathbb{E} \|J_{d_N}(U^* D(\vec{\lambda}) U)\|^2 = -\Delta \hat{\mu}_{\vec{\lambda}}(X)|_{X=0}, \\ m_4(\mathcal{P}_{\vec{\lambda}_N}) := \mathbb{E} \|J_{d_N}(U^* D(\vec{\lambda}) U)\|^4 = \Delta^2 \hat{\mu}_{\vec{\lambda}}(X)|_{X=0}, \end{cases}$$

We translate $\vec{\lambda}$ by its center of mass to make the center of mass of $\mathcal{P}_{\vec{\lambda}}$ equal to 0, i.e. $\sum \lambda_j = 0$. Using a formula for $\hat{\mu}_{\vec{\lambda}}(X)$ in terms of Schur polynomials, we prove

LEMMA 2. *Let p_k be the power functions (0.6). Assume that $p_1(\vec{\lambda}) = 0$. Then,*

$$\begin{cases} \Delta \hat{\mu}_{\vec{\lambda}}(0) = \frac{p_2(\vec{\lambda})}{d_N+1}, \\ \Delta^2 \hat{\mu}_{\vec{\lambda}}(0) = \beta_4(d_N) p_2^2(\vec{\lambda}), \\ \text{with } \beta_4(d_N) = \left(\frac{4d_N(d_N-1)}{(d_N+1)d_N^2(d_N-1)} - \frac{4d_N(d_N-1)}{(d_N+2)(d_N+1)d_N(d_N-2)} + \frac{(12d_N^2+4d_N(d_N-1))}{(d_N+3)(d_N+2)(d_N+1)d_N} \right). \end{cases}$$

The proof of Theorem 1 follows directly from Lemma 2 and the Kolmogorov SLLN (strong law of large numbers). When d_N grows fast enough it also follows directly from the Borel-Cantelli Lemma. We first introduce notation for the basic random variables:

Definition:

$$\begin{cases} Y_N^A : \mathcal{ONB}_N \rightarrow [0, +\infty), & \Psi = (U_{d_1}, U_{d_2}, \dots) \\ Y_N^A(\Psi) := \|J_{d_N}(U_N^* D(\vec{\lambda}) U_N) - \overline{D(\vec{\lambda})}\|^2 = \|J_{d_N}(U_N^* D_0(\vec{\lambda}) U_N)\|^2 \end{cases}$$

Then Lemma 1-Lemma 2 determine the asymptotics of their mean and variance

$$\text{Var}(Y_N^A) := \mathbb{E}((Y_N^A)^2) - (\mathbb{E}(Y_N^A))^2.$$

COROLLARY 1.

$$\text{Var}(Y_N^A) = \left(\beta_4(d_N) - \frac{1}{(d_N+1)^2} \right) p_2^2(\vec{\Lambda}) \simeq \frac{3}{d_N^2} p_2^2(\vec{\Lambda}).$$

The Lemma first implies that $\mathbb{E}\|J_{d_N}(U^* D(\vec{\lambda}) U)\|^2$ is bounded for all $A \in \Psi^0(M)$. Hence, $\mathbb{E}(\frac{1}{d_N}\|J_{d_N}(U^* D(\vec{\lambda}) U)\|^2) \rightarrow 0$ as long as $d_N \rightarrow \infty$. Thus, the mean of the quantum variances (0.2) tends to zero. As in [Z1, SZ] we then apply the Kolmogorov SLLN (or the martingale convergence theorem). The $\{Y_N^A\}$ is a sequence of independent random variables as N varies and Lemma 2 shows that they have bounded variance. Hence the SLLN implies that the partial sums,

$$S_N := \sum_{n \leq N} \frac{1}{d_n} (Y_n^A - \mathbb{E}Y_n^A) \quad (0.11)$$

have the property,

$$\frac{1}{N} S_N \rightarrow 0, \text{ almost surely} \quad (0.12)$$

and this is equivalent to quantum ergodicity of random orthonormal bases. As mentioned above, if d_N grows at a faster rate one can obtain stronger results from the Borel-Cantelli Lemma: E.g. if $\sum_{n=1}^{\infty} \frac{1}{d_n} < \infty$, one obtains almost sure convergence $\frac{1}{d_n} Y_n^A \rightarrow 0$ (a.s.).

Since the argument above only requires that $\frac{1}{d_n} \mathbb{E}Y_n^A \rightarrow 0$ and $\text{Var}(Y_n^A)$ is bounded, it does not require any assumption that $\mathbb{E}Y_n^A$ tends to a limit. Our calculations therefore go beyond what is necessary for almost sure quantum ergodicity, and pertain to the asymptotic geometry of the polytopes $\mathcal{P}_{\vec{\lambda}}$. There is a natural condition on this sequence of polytopes:

Definition: We say that the sequence $\{\mathcal{H}_N\}$ has Szegő asymptotics if, for all $A \in \Psi^0(M)$, there exists a unique weak* limit, $\nu_{\vec{\lambda}_N} \rightarrow \nu_A \in \mathcal{M}(\mathbb{R})$ as $N \rightarrow \infty$. Here, $\mathcal{M}(\mathbb{R})$ is the set of probability measures on \mathbb{R} .

Under this stronger assumption, Lemma 2 gives moment asymptotics:

PROPOSITION 1. *Let $\vec{\lambda}_N \in \mathbb{R}^{d_N}$ be a sequence of vectors with the property that the empirical measures (0.5) tend to a weak limit ν . Then*

$$\begin{cases} m_2(\mathcal{P}_{\vec{\lambda}_N}) \rightarrow \int_{\mathbb{R}} (t - \bar{t})^2 d\nu, \\ m_4(\mathcal{P}_{\vec{\lambda}_N}) \rightarrow 4 \left(\int_{\mathbb{R}} (t - \bar{t})^2 d\nu \right)^2 \end{cases}$$

This Proposition is closely related to the “Weingarten theorem” that the matrix elements $\sqrt{d_N}U_{ij}$ are asymptotically complex normal random variables, where U_{ij} are the matrix elements of $U \in U(d_N)$ [W]. Perhaps this explains why the fourth moment is a constant multiple of the square of the second moment. It would be interesting to see if the pattern continues; we plan to study \mathcal{P}_λ further in [Z3].

0.1. Discussion. The motivation for proving quantum ergodicity of random orthonormal bases for \mathcal{H}_N of any dimensions tending to infinity was prompted by the general question: how many diffuse states (modes or quasi-modes) does it take to synthesize localized modes or quasi-modes? Vice-versa, how many localized states does it take to synthesize diffuse states? We would like to synthesize entire orthonormal bases rather than individual states and measure the dimensions of the space of states in terms of the Planck constant \hbar . Let us consider some examples.

In the case of the standard S^2 , the eigenspaces \mathcal{H}_N of Δ are the spaces of spherical harmonics of degree N . They have the well-known highly localized basis Y_m^N of joint eigenfunctions of Δ and of rotations around the x_3 -axis. By localized we mean that a sequence $\{Y_m^N\}$ with $m/N \rightarrow \alpha$ microlocally concentrates on the invariant tori in S^*S^2 where $p_\theta = \alpha$. Here, $p_\theta(x, \xi) = \xi(\frac{\partial}{\partial \theta})$ where $\frac{\partial}{\partial \theta}$ generates the x_3 -axis rotations. On the other hand, it is proved in [Z1] that independent “random” orthonormal bases of \mathcal{H}_N are quantum ergodic, i.e. are highly diffuse in S^*S^2 . Since $\dim \mathcal{H}_N = 2N + 1$, it is perhaps not surprising that the same eigenspace can have both highly localized and highly diffuse orthonormal bases when its dimension is so large. The question is, how large must it be for such incoherently related bases to exist?

A setting where the eigenvalues have high multiplicity but of a lower order of magnitude than on S^2 is that of flat rational tori \mathbb{R}^n/L such as $\mathbb{R}^n/\mathbb{Z}^n$. Of course it has an orthonormal basis of localized eigenfunctions, $e^{i\langle k, x \rangle}$. The key feature of such rational tori is the high multiplicity of eigenvalues of the Laplacian Δ of the flat metric. It is well-known and easy to see that the multiplicity is the number of lattice points of the dual lattice L^* lying on the surface of a Euclidean sphere. We denote the distinct multiple Δ -eigenvalues by μ_N , the corresponding eigenspace by \mathcal{H}_N and the multiplicity of μ_N^2 by $d_N = \dim \mathcal{H}_N$. In dimensions $n \geq 5$, $d_N \sim \mu_N^{n-2}$, one degree lower than the maximum possible multiplicity of a Δ -eigenvalue on any compact Riemannian manifold, achieved on the standard S^n . Further, $\frac{1}{d_N} \text{Tr} \Pi_N A \Pi_N \rightarrow \omega(A)$. Hence, the results of this article show that despite the relatively slow growth of d_N on a flat rational torus, orthonormal bases of \mathcal{H}_N in dimensions ≥ 5 are almost surely quantum ergodic. The statement for dimensions 2, 3, 4 is more complicated (see §4.1).

An interesting setting where the behavior of eigenfunctions is largely unknown is that of KAM systems. For these, one may construct a ‘nearly’ complete and orthonormal basis for $L^2(M)$ by highly localized quasi-modes associated to the Cantor set of invariant tori. It seems unlikely that the actual eigenfunctions are quantum ergodic; but the results of this article show that if they resemble random combinations of the quasi-mode, then it is possible that they are. Further discussion is in §4.2.

1. BACKGROUND

In this section, we review the definition of random orthonormal basis and relate it to properties of the moment map for the diagonal action of the maximal torus T_{d_N} on coadjoint orbits of $U(d_N)$.

1.1. Random orthonormal bases of eigenspaces. Suppose that we have a sequence of Hilbert spaces \mathcal{H}_N $N = 1, 2, \dots$ of dimensions $d_N = \dim \mathcal{H}_N \rightarrow \infty$. We define the large Hilbert space

$$\mathcal{H} = \bigoplus_{N=1}^{\infty} \mathcal{H}_N$$

and orthogonal projections

$$\Pi_N : \mathcal{H} \rightarrow \mathcal{H}_N. \quad (1.1)$$

We then consider the orthonormal bases (0.4) of \mathcal{H} which arise from sequences of orthonormal bases of \mathcal{H}_N .

1.2. The basic random variables. Let $A \in \Psi^0(M)$ be a zeroth order pseudo-differential operator. By a Toeplitz operator we mean the compression T_N^A (0.1) of A to \mathcal{H}_N .

Given one ONB of \mathcal{H}_N , T_N^A can be identified with a Hermitian $d_N \times d_N$ matrix. We fix orthonormal bases $\{e_j^N\}_{j=1}^{d_N}$ of \mathcal{H}_N and introduce the random variables:

$$A_{Nj}(\Psi) = |\langle A\psi_j^N, \psi_j^N \rangle - \omega(A)|^2 = |(T_N^A \psi_j^N, \psi_j^N) - \omega(A)|^2 = |(U_N^* T_N^A U_N e_j^N, e_j^N) - \omega(A)|^2, \quad (1.2)$$

where $\Psi = \{U_N\}$, $U_N \in U(d_N) \equiv \mathcal{OBN}_N$. We also define

$$\widehat{A}_{Nj}(\Psi) = \left| (U_N^* T_N^A U_N e_j^N, e_j^N) - \frac{1}{d_N} \text{Tr } T_N^A \right|^2. \quad (1.3)$$

Evidently,

$$\frac{1}{d_N} Y_N^A(\Psi) = \frac{1}{d_N} \sum_{j=1}^{d_N} \widehat{A}_{Nj}(\Psi) = \frac{1}{d_N} \sum_{j=1}^{d_N} A_{Nj}(\Psi) + o(1) \quad (1.4)$$

(where the $o(1)$ term is independent of Ψ). Thus,

LEMMA 1.1. [Z1, SZ] *The ergodic property of an ONB Ψ (\mathcal{EP}) is equivalent to:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \frac{1}{d_n} Y_n^A(\Psi) = 0, \quad \forall A \in \Psi^0(M). \quad (1.5)$$

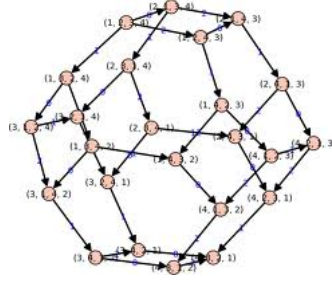
As mentioned in the introduction, it follows by a standard diagonal argument that almost all the individual elements $\langle A\psi_{N,j}, \psi_{N,j} \rangle$ tend to $\omega(A)$ for all A . We do not discuss this step since it is nothing new.

1.3. Moment map interpretation. In the case where the components of $\vec{\lambda}_N$ are distinct, the convex polytope $\mathcal{P}_{\vec{\lambda}_N}$ is the permutahedron determined by λ , that is, the simple convex polytope defined as the convex hull of the points $\{\sigma(\vec{\lambda}_N)\}$ where $\sigma \in S_N$ runs over the symmetric group on d_N letters (i.e. the Weyl group of $U(d_N)$). The center of mass is the unique point $X \in \mathcal{P}_{\vec{\lambda}_N}$ so that

$$\sum_{\sigma \in S_{d_N}} \overline{X\sigma(\vec{\lambda}_N)} = 0 \iff X = \frac{1}{(d_N)!} \sum_{\sigma \in S_{d_N}} \sigma(\vec{\lambda}_N)$$

where $\overline{XY} = X - Y$ is the vector from X to Y . The center of mass is evidently invariant under S_{d_N} , hence has the form (a, a, \dots, a) for some a and clearly $a = \frac{1}{d_N} \sum_{j=1}^{d_N} \lambda_j$.

In effect, we want to asymptotically calculate the moments of inertia of the sequence of permutahedra associated to a Toeplitz operator.



Permutahedron

1.4. Symmetric polynomials and Schur polynomials. The elementary symmetric polynomials of N variables are defined by

$$e_k(X_1, \dots, X_N) = \sum_{i_1 < i_2 < \dots < i_k \leq N} x_{i_1} \cdots x_{i_k}.$$

If one replaces $<$ by \leq one obtains the complete symmetric polynomials h_k . The Schur polynomials are symmetric polynomials defined by

$$S_\lambda = \det(h_{\lambda_i + j - i}) = \det(e_{\mu_i + j - i})$$

where μ is a dual partition to λ .

1.5. Fourier transform of the orbit. We can compute the moments using the Fourier transform (0.10) of the orbital measure on the orbit of $D(\vec{\lambda})$.

An explicit formulae for $\hat{\mu}_\lambda(X)$ is given in the first line of the proof of Theorem 5.1 of [OV]:

LEMMA 1.2. For $U(d)$,

$$\hat{\mu}_{\vec{\lambda}}(X) = (d-1)! \cdots 0! \sum_{\mu: \ell(\mu) \leq d} \frac{S_\mu(X) S_\mu(i\vec{\lambda})}{(\mu_1 + d - 1)! (\mu_2 + d - 2)! \cdots \mu_N!}. \quad (1.6)$$

Here, $\ell(\mu)$ is the number of rows of the partition μ . The degree of s_μ is $|\mu|$, the number of boxes.

Since we would like to shift the center of mass of $\mathcal{P}_{\vec{\lambda}}$ to the origin, we mainly consider $\hat{\mu}_{\vec{\lambda}}(X)$ the Fourier transform of the traceless orbit (see (0.9)).

2. PROOF OF PROPOSITION 1: MOMENT ASYMPTOTICS

2.1. Second moment asymptotics. We now prove:

LEMMA 3. [Z1, Z2, SZ] *Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_{d_N}) \in \mathbb{R}^{d_N}$, and let $D_0(\vec{\lambda}_N)$ denote the trace zero diagonal matrix with entries (0.9). Thus, $p_1(\vec{\lambda}) = 0$ (0.9). Then*

$$\mathbb{E}Y_N^A = \int_{U(d)} \|J_{d_N}(U^* D_0(\vec{\lambda}) U)\|^2 dU = \frac{p_2(\vec{\lambda})}{d_N + 1}, \quad (2.1)$$

where as above, dU is the normalized Haar probability measure on $U(d_N)$.

This Lemma was proved in [Z1, Z2, SZ] using the so-called Itzykson-Zuber-Harish-Chandra formula for the Fourier transform of the orbit, and again using Gaussian integrals. The proof we give here generalizes better to higher moments. We also sketch a proof using the Weingarten formulae.

Proof. We use Lemma 1.2 to obtain

$$\mathbb{E} \|J_{d_N}(U^* D(\vec{\lambda}) U)\|^2 = (d_N - 1)! \cdots 0! \sum_{\mu: |\mu|=2, \ell(\mu) \leq d_N} \frac{\Delta S_\mu(0) S_\mu(i\vec{\lambda})}{(\mu_1 + d_N - 1)! (\mu_2 + d_N - 2)! \cdots \mu_{d_N}!}.$$

We sum over the Young diagrams with exactly two boxes and $\leq d_N$ rows. There are just two of them: one row of two boxes or two rows of one box each corresponding respectively to the Schur functions $S_{(2,0)}$, $S_{(1,1)}$. Note that $S_{1^k} = e_k$ is the k th elementary symmetric function and $S_{(k)} = h_k$ is the complete k th degree symmetric function.

We then translate $\vec{\lambda}$ to $\vec{\Lambda}$ so that $\sum_j \Lambda_j = e_1(\vec{\Lambda}) = 0$, i.e. we replace $D(\vec{\lambda}_N)$ by $D_0(\vec{\lambda}_N)$.

Since the degree $|\mu| = 2$, then we can only use $\mu = (2)$, (11) and

$$S_{(1,1)} = e_2 = \sum_{i < j} x_i x_j, \quad S_{(2,0)} = e_1^2 - e_2 = \sum_j x_j^2 + \sum_{i < j} x_i x_j.$$

But

$$\Delta e_2 \equiv 0, \quad \Delta(e_1^2 - e_2) = 2 \|\nabla e_1\|^2 = 2d_N.$$

For each monomial $X_i X_j$ we have $\Delta X_i X_j = 2\delta_{ij}$. Thus, $\Delta S_{(1,1)} = 0$ and $\Delta S_{(2,0)} = 2d_N$. Since the Schur polynomials are homogeneous of degree 2, we can remove the i under the Schur polynomials to get an overall factor of -1 , which is cancelled by the $-$ sign from Δ . Thus,

$$\begin{aligned} \mathbb{E} \|J_{d_N}(U^* D_0(\vec{\lambda}) U)\|^2 &= (2d_N)(d_N - 1)! \frac{S_{(2,0)}(i\vec{\Lambda})}{(d_N + 1)!} = \frac{(2d_N)}{(d_N + 1)d_N} S_{(2,0)}(i\vec{\Lambda}) \\ &= \frac{2}{d_N + 1} S_{(2,0)}(\vec{\Lambda}_N) = \frac{2}{d_N + 1} (e_1^2 - e_2)(\vec{\Lambda}). \end{aligned}$$

Since $e_1(\vec{\Lambda}_N) = 0$ we find that

$$\mathbb{E} \|J_{d_N}(U^* D_0(\vec{\lambda}) U)\|^2 = -\frac{2}{d_N + 1} e_2(\vec{\Lambda}) = \frac{1}{d_N + 1} p_2(\vec{\Lambda}_N).$$

Here we use that

$$e_1 = p_1, \quad 2e_2 = e_1 p_1 - p_2.$$

The formula agrees with the one stated in the Lemma 3. □

2.2. Weingarten formulae for the expectation. As a second proof, we use the Weingarten formula for integrals of polynomials over $U(N)$ [W]. We denote the eigenvalues of $D_0(\vec{\lambda})$ by $\vec{\Lambda}$. Then,

$$||\text{diag}(U^* D_0(\vec{\lambda}) U)||^2 = \sum_{j_1, j_2} \Lambda_{j_1} \Lambda_{j_2} \sum_i |U_{ij_1}|^2 |U_{ij_2}|^2.$$

The Weingarten formulae for these special polynomials state that asymptotically $\sqrt{d_N} |U_{ij}|^2$ is a complex Gaussian random variable of mean zero and variance one. Thus, to leading order,

$$\int_{U(d_N)} |U_{ij_1}|^2 |U_{ij_2}|^2 dU \simeq d_N^{-2} (1 + \delta_{j_1 j_2}),$$

and

$$\sum_{j_1, j_2} \Lambda_{j_1} \Lambda_{j_2} \sum_i \int_{U(d_N)} |U_{ij_1}|^2 |U_{ij_2}|^2 dU \simeq d_N^{-1} (2 \sum_j \Lambda_j^2 + \sum_{j_1 \neq j_2} \Lambda_{j_1} \Lambda_{j_2}). \quad (2.2)$$

Since

$$0 = (\sum \Lambda_j)^2 = \sum \Lambda_j^2 + \sum_{j \neq k} \Lambda_j \Lambda_k$$

we get

$$(2.2) \simeq d_N^{-1} \sum_j \Lambda_j^2.$$

2.3. Proof of Proposition 1: Variance and fourth moment asymptotics. We now prove the 4th moment identity in Proposition 1, which is the main new step in this article.

To calculate the variance of Y_N^A we use the expression in Lemma 1 in terms of $\hat{\mu}_{\vec{\lambda}}$ and then use the formula of Lemma 1.2.

A Schur polynomial $S_{n_1, \dots, n_d}(x_1, \dots, x_d)$ of degree n in d variables is parameterized by a partition of the degree $n = n_1 + n_2 + \dots + n_d$ into d parts. When $n = 4$ and $d \geq 4$ there are 5 partitions:

$$\left\{ \begin{array}{l} S_{1,1,1,1}(x) = e_4 = \sum_{1 \leq i < j < k < \ell} x_i x_j x_k x_\ell; \\ S_{2,1,1}(x_1, \dots, x_N) = e_1 e_3 \\ S_{2,2,0} = e_2^2 - e_1 e_3 \\ S_{4,0,0} = e_1^4 - 3e_1^2 e_2 + 2e_1 e_3 + e_2^2 \\ S_{3,1,0} = e_1^2 e_2 - e_2^2 - e_1 e_3. \end{array} \right. \quad (2.3)$$

We note that $\Delta e_k(X) = 0$ for all k , so $\Delta e_k e_n = 2 \nabla e_k \cdot \nabla e_n$. Also, ∇e_1 is a constant vector. So $\Delta^2 e_1 e_3 = \nabla e_1 \cdot \nabla \Delta e_3 = 0$ and

$$\Delta^2 e_1^2 e_2 = 4 \Delta(e_1 \nabla e_1 \cdot \nabla e_2) = 8 \nabla e_1 \cdot \nabla(\nabla e_1 \cdot \nabla e_2) = 8 \text{TrHesse}_2 = 0.$$

Here, Hess denotes the Hessian. We also use that $\Delta(\nabla f \cdot \nabla g) = 2\text{Hess}(f) \cdot \text{Hess}(g)$ when $\Delta f = \Delta g = 0$. Also,

$$\nabla e_1 \cdot \nabla(\nabla e_1 \cdot \nabla e_2) = (1, 1, \dots, 1) \cdot \sum_{j,k} \frac{\partial^2 e_2}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k} = \text{Tr Hess}(e_2) = 0.$$

Then,

$$\Delta^2 e_2^2 = 2\Delta(\nabla e_2 \cdot \nabla e_2) = 4\|\text{Hess}(e_2)\|^2 = 4d_N(d_N - 1).$$

Further, $\Delta e_1^2 = 2\nabla e_1 \cdot \nabla e_1 = 2d_N$, so that

$$\Delta^2 e_1^4 = \Delta(2(\Delta e_1^2)e_1^2 + 2\nabla e_1^2 \cdot \nabla e_1^2) = \Delta(4d_N e_1^2 + 2e_1^2 d_N) = 12d_N^2.$$

We recall Newton's identities,

$$\begin{cases} e_1 = p_1 \\ 2e_2 = e_1 p_1 - p_2 \\ 3e_3 = e_2 p_1 - e_1 p_2 + p_3 \\ 4e_4 = e_3 p_1 - e_2 p_2 + e_1 p_3 - p_4. \end{cases}$$

We note that $\Delta e_k \equiv 0$ for all k , so at $X = 0$,

$$\begin{cases} \Delta^2 S_{1,1,1,1} = \Delta e_4 \equiv 0; \\ \Delta S_{2,1,1} = \Delta^2 e_1 e_3 = 2\Delta(\nabla e_1 \cdot \nabla e_3) = \nabla e_1 \cdot \nabla \Delta e_3 = 0; \\ \Delta^2 S_{2,2,0} = \Delta^2 e_2^2 = 2\Delta(\nabla e_2 \cdot \nabla e_2) = 4\|\text{Hess}(e_2)\|^2 = 4d_N(d_N - 1) \\ \Delta S_{4,0,0} = \Delta^2 e_1^4 - (3)8\nabla e_1 \cdot \nabla(\nabla e_1 \cdot \nabla e_2) + 4\|\text{Hess}(e_2)\|^2 = 12d_N^2 + 4d_N(d_N - 1) \\ \Delta S_{3,1,0} = 8\nabla e_1 \cdot \nabla(\nabla e_1 \cdot \nabla e_2) - \|\text{Hess}(e_2)\|^2 = -4\|\text{Hess}(e_2)\|^2 = -4d_N(d_N - 1). \end{cases} \quad (2.4)$$

By routine calculations and Lemma 1.2 we have,

$$\begin{aligned} \Delta^2 \hat{\mu}_{\vec{\Lambda}}(0) &= (d_N - 1)! \cdots 0! \sum_{\mu: |\mu|=4} \frac{\Delta^2 S_{\mu}(0) S_{\mu}(i\vec{\Lambda})}{(\mu_1 + d_N - 1)!(\mu_2 + d_N - 2)! \cdots \mu_{d_N}!} \\ &= (d_N - 1)!(d_N - 2)! \frac{\Delta^2 S_{2,2,0}(0) S_{2,2,0}(i\vec{\Lambda})}{(d_N + 1)!(d_N)!} \\ &\quad + (d_N - 1)! \frac{\Delta^2 S_{4,0,0}(0) S_{4,0,0}(i\vec{\Lambda})}{(d_N + 3)!} \\ &\quad + (d_N - 1)!(d_N - 2)! \frac{\Delta^2 S_{3,1,0}(0) S_{3,1,0}(i\vec{\Lambda})}{(d_N + 2)!(d_N - 1)!} \\ &= \frac{\Delta^2 S_{2,2,0}(0) S_{2,2,0}(i\vec{\Lambda})}{(d_N + 1)d_N^2(d_N - 1)} \\ &\quad + \frac{\Delta^2 S_{4,0,0}(0) S_{4,0,0}(i\vec{\Lambda})}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} + \frac{\Delta^2 S_{3,1,0}(0) S_{3,1,0}(i\vec{\Lambda})}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. \end{aligned} \quad (2.5)$$

By (2.4), we then have

$$\Delta^2 \hat{\mu}_{\vec{\Lambda}}(0) = \frac{4d_N(d_N-1)S_{2,2,0}(i\vec{\Lambda})}{(d_N+1)d_N^2(d_N-1)} + \frac{(12d_N^2+2d_N(d_N-1))S_{4,0,0}(i\vec{\Lambda})}{(d_N+3)(d_N+2)(d_N+1)d_N} + \frac{-4d_N(d_N-1)S_{3,1,0}(i\vec{\Lambda})}{(d_N+2)(d_N+1)d_N(d_N-2)}. \quad (2.6)$$

Recalling (2.3) and that $e_1(\vec{\Lambda}) = 0$, we get

$$\Delta^2 \hat{\mu}_{\vec{\Lambda}}(0) = \frac{4d_N(d_N-1)e_2^2(i\vec{\Lambda})}{(d_N+1)d_N^2(d_N-1)} + \frac{(12d_N^2+4d_N(d_N-1))e_2^2(i\vec{\Lambda})}{(d_N+3)(d_N+2)(d_N+1)d_N} + \frac{-4d_N(d_N-1)e_2^2(i\vec{\Lambda})}{(d_N+2)(d_N+1)d_N(d_N-2)}. \quad (2.7)$$

Further recalling that $2e_2 = e_1p_1 - p_2$ we finally get

$$\Delta^2 \hat{\mu}_{\vec{\Lambda}}(0) = \frac{d_N(d_N-1)p_2^2(i\vec{\Lambda})}{(d_N+1)d_N^2(d_N-1)} + \frac{(3d_N^2+d_N(d_N-1))p_2^2(i\vec{\Lambda})}{(d_N+3)(d_N+2)(d_N+1)d_N} + \frac{-d_N(d_N-1)p_2^2(i\vec{\Lambda})}{(d_N+2)(d_N+1)d_N(d_N-2)}. \quad (2.8)$$

Since the polynomials are homogeneous of degree 4, the factor of i inside the polynomials may be removed, and we get

$$\mathbb{E}||J_{d_N}(U^*D_0(\vec{\Lambda})U)||^4 = \frac{d_N(d_N-1)p_2^2(\vec{\Lambda})}{(d_N+1)d_N^2(d_N-1)} + \frac{(3d_N^2+d_N(d_N-1))p_2^2(\vec{\Lambda})}{(d_N+3)(d_N+2)(d_N+1)d_N} + \frac{-d_N(d_N-1)p_2^2(\vec{\Lambda})}{(d_N+2)(d_N+1)d_N(d_N-2)}.$$

As $N \rightarrow \infty$ the leading asymptotics of the outer terms cancel and the middle term is asymptotic to $\frac{4}{d_N^2}p_2^2(\vec{\Lambda}_N)$. We note that $\frac{p_2(\vec{\Lambda}_N)}{d_N}$ is bounded. If the empirical measure of eigenvalues tends to a limit measure, then $\frac{p_2(\vec{\Lambda}_N)}{d_N}$ tends to its second moment.

Together with Lemma 2.1, this completes the proof of Proposition 1. Corollary 1 follows by subtracting the square of the expectation.

3. COMPLETION OF PROOF OF THEOREM 1

By the assumption of Definition (0.7)

$$\omega(A) = \frac{1}{d_N} \text{Tr } T_N^A + o(1), \quad (3.1)$$

By Lemma 1-Corollary 1, the variances of the independent random variables $\frac{1}{d_N}Y_N^A$ are bounded. Hence, as explained in the introduction (see also [Z1, SZ]), (1.5) follows from Lemma 1-Corollary 1 and the Kolmogorov strong law of large numbers, which gives

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\frac{1}{d_n} Y_n^A \right) = 0 \quad \text{almost surely.} \quad (3.2)$$

By (1.4),

$$\sup_{\mathcal{O}\mathcal{NB}_N} |X_N^A - \frac{1}{d_N} Y_N^A| = o(1).$$

Hence also

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\frac{1}{d_n} X_n^A \right) = 0 \quad \text{almost surely.} \quad (3.3)$$

If the dimensions d_N grow fast enough so that $\frac{1}{d_N}$ is summable, then we obtain a stronger form from the fact that $\sum_{n=1}^{\infty} \mathbb{E} \frac{1}{d_n} Y_n^A$ is finite hence the general term must tend to zero almost everywhere. It follows again that $\mathbb{E} \frac{1}{d_n} X_n^A \rightarrow 0$ almost everywhere.

4. APPLICATIONS

4.1. **Fat tori.** Theorem 1 applies to eigenspaces of the Laplacian on the flat torus $\mathbb{R}^d/\mathbb{Z}^d$ (or other rational lattices) of dimension ≥ 5 and for many eigenspaces in dimensions $d = 2, 3, 4$.

PROPOSITION 4.1. *Random orthonormal bases of Δ -eigenspaces of the flat torus $\mathbb{R}^d/\mathbb{Z}^d$ are quantum ergodic for $d \geq 5$. Also for $d = 2, 3, 4$ for special eigenspaces (specified below).*

The only condition on the eigenspaces for Theorem 1 is that (0.7) holds, and we now recall the known results on this problem. Given $A \in \Psi^0$, we denote the eigenspaces on a flat torus, enumerated in order of the eigenvalue by \mathcal{H}_N and by Π_N the orthogonal projection to \mathcal{H}_N ¹.

LEMMA 4.2. *The condition (0.7) is valid in dimensions ≥ 5 on $\mathbb{R}^d/\mathbb{Z}^d$. That is,*

$$\frac{1}{d_N} \text{Tr} A \Pi_N \sim \int_{S^*T^m} a(x, \omega) dx \wedge d\omega.$$

It follows that $\frac{1}{d_N} Y_N^A \rightarrow 0$ almost surely.

In dimensions 2, resp. 3, resp. 4 there are restrictions on the sequence of eigenvalues given in [EH], resp. [DSP], resp. [P]. For eigenvalues in the allowed sequences, (0.7) is valid.

Proof. We use the basis $e_k^N = e^{i\langle k, x \rangle}$ with $|k| = \mu_N$. Then

$$\langle A e_k^N, e_k^N \rangle = \int_{\mathbb{R}^n/\mathbb{Z}^n} \sigma_A(x, k) dx.$$

Hence

$$\frac{1}{d_N} \text{Tr} \Pi_N A = \sum_{k: |k| = \mu_N} \int_{\mathbb{R}^n/\mathbb{Z}^n} \sigma_A(x, k) dx.$$

In dimensions $n \geq 5$, $d_N \sim \mu_N^{n-2}$. It is proved that lattice points of fixed norm on a sphere of radius \sqrt{n} become uniformly distributed as $n \rightarrow \infty$ [P]. It follows that

$$\frac{1}{d_N} \sum_{k: |k| = \mu_N} \int_{\mathbb{R}^n/\mathbb{Z}^n} \sigma_A(x, k) dx \rightarrow \int_{S^*T^m} a(x, \omega) dx \wedge d\omega.$$

As in the last step of the proof of Theorem 1, $\mathbb{E} \frac{1}{d_N} X_N^A$ is summable when $n \geq 5$.

The Liouville limit formula is true in dimension 4 when the number of lattice points grows linearly in n . The condition on n is given in [P]. In dimension 3, the equidistribution result is proved in [DSP] with similar conditions on the sequence of integers n .

Dimension 2 is more complicated. In dimension 2, the eigenvalues of integers n for which there exist lattice points (a, b) on the circle $a^2 + b^2 = n$. It is necessary that all prime factors of n are congruent to 1 modulo 4. In [EH] it is shown that for almost all such n , the lattice points on the circle become uniformly distributed as $n \rightarrow \infty$. \square

Remark: In the case of a generic lattice $L \subset \mathbb{R}^d$, the multiplicity of eigenvalues of Δ on \mathbb{R}^d/L is two. The analogue of the eigenspaces above are spectral subspaces for $\sqrt{\Delta}$ of shrinking width w . Thus, one considers the exponentials $e^{i\langle \ell, x \rangle}$ for $\ell \in L$ with $|\ell| \in [\lambda - Cw, \lambda + w]$. It follows from the lattice point results of [G] that in dimensions $d \geq 5$, the number of eigenvalues of

¹Thanks to Z. Rudnick for explanations and references

an irrational flat torus in $[\lambda, \lambda + O(\lambda^{-1})]$ is of order λ^{d-2} . The question whether the trace asymptotics (0.7) hold for the span of the corresponding eigenfunctions does not appear to have been studied.

4.2. Quasi-modes. Theorem 1 is not restricted to eigenspaces of the Laplacian and is equally valid for spaces of quasi-modes. We refer to [CV, Po] for background on quasi-modes. Following [Po], we define a C^∞ quasimode of infinite order for $\hbar^2\Delta$ with index set \mathcal{M}_\hbar to be a family

$$\mathcal{Q} = \{(\psi_m(\cdot, \hbar), \mu_m(\hbar)) : m \in \mathcal{M}_\hbar\}$$

of approximate eigenfunctions satisfying

$$\begin{cases} (i) \|(\hbar^2\Delta - \mu_m(\hbar))\psi_m(\cdot, \hbar)\|_{H^s} = O_M(\hbar^M), & (\forall M \in \mathbb{Z}^+), \\ (ii) |\langle \psi_m, \psi_n \rangle - \delta_{mn}| = O_M(\hbar^M), & (\forall M \in \mathbb{Z}^+). \end{cases} \quad (4.1)$$

It follows by the spectral theorem that for any $M \in \mathbb{Z}^+$, there exists at least one eigenvalue of $\hbar^2\Delta$ in the interval

$$I_{mM}^\hbar = [\mu_m(\hbar) - \hbar^M, \mu_m(\hbar) + \hbar^M],$$

and

$$\|E_{I_{mM}^\hbar} \psi_k - \psi_k\|_{H^s} = O_M(\hbar^M). \quad (4.2)$$

Here, E_I denotes the spectral projection for $\hbar^2\Delta$ corresponding to the interval I . We denote the quasi-classical eigenvalue spectrum of $\hbar\sqrt{\Delta}$ by

$$QSp_\hbar = \{\mu_m(\hbar) : m \in \mathcal{M}_\hbar\}.$$

Since quasi-eigenvalues $\mu_m(\hbar)$ are only defined up to errors of order \hbar^∞ , there is a notion of ‘multiple quasi-eigenvalue’ defined as follows: we say $\mu_m(\hbar) \sim \mu_n(\hbar)$ if $\mu_m - \mu_n = O(\hbar^\infty)$ and define the multiplicity of $\mu_m(\hbar)$ by

$$\text{mult}(\mu_m(\hbar)) = \#\{n : \mu_m(\hbar) \sim \mu_n(\hbar)\} = \dim \text{Span}\{\psi_n(\cdot, \hbar) : (\hbar^2\Delta - \mu_m(\hbar))\psi_n = O(\hbar^\infty)\}.$$

We then introduce slightly larger intervals $\mathcal{I}_{m,\hbar}$ (if need be) so that

$$QSp(\hbar) \subset \bigcup_{m \in \mathcal{M}'} \mathcal{I}_{m,\hbar}, \quad \mathcal{I}_{m,\hbar} \cap \mathcal{I}_{n,\hbar} = \emptyset \quad (m \neq n).$$

Here, \mathcal{M}' consists of equivalence classes of indices (corresponding to equivalence classes of quasimodes). We denote by \mathcal{H}_m^\hbar the span of the quasimodes $\{\psi_m(\cdot, \hbar) : \mu_m(\hbar) \in \mathcal{I}_m^\hbar\}$. Then

$$\|E_{\mathcal{I}_m^\hbar} v - v\| = O(\hbar^\infty), \quad \text{if } v \in \mathcal{H}_m^\hbar.$$

Theorem 1 applies to quasi-mode spaces \mathcal{H}_m^\hbar as long as their dimensions tend to infinity and as long as there exists a unique limit state for $\frac{1}{\dim \mathcal{H}_m^\hbar} \text{Tr} A|_{\mathcal{H}_m^\hbar}$. One might expect true modes (eigenfunctions) with eigenvalues in the intervals I_{mM}^\hbar to be close to linear combinations of the quasi-modes with quasi-eigenvalues in that interval. The question raised by Theorem 1 is whether they behave like random linear combinations or not. If they do, Theorem 1 gives their quantum limits.

In particular, this bears on the question whether Δ -eigenfunctions of compact Riemannian manifolds (M, g) with KAM geodesic flow might be quantum ergodic. It seems unlikely that they are, but we are not aware of a proof that they are not. For such KAM (M, g) , a

large family of quasi-modes is constructed in [CV, Po] which localize on the invariant tori of the KAM Cantor set of tori. Without reviewing the results in detail, the ‘large’ family has positive spectral density, i.e. the number of quasi-eigenvalues $\leq \mu$ grows like a positive constant times μ^n where $n = \dim M$.

To our knowledge, the multiplicities and trace asymptotics for KAM quasi-modes have not been studied at this time. As in the discussion of flat tori, one would need to determine the equidistribution law of the tori in the invariant Cantor set corresponding to eigenvalues (or pseudo-eigenvalues) of $\sqrt{\Delta}$ in very short intervals $I_\lambda = [\lambda - w, \lambda + w]$. The orthonormal basis of eigenfunctions is not simple to relate to the near orthonormal basis of quasi-modes in this case, but we might expect that a positive density of the eigenfunctions are mainly given as linear combinations of KAM quasi-modes with quasi-eigenvalues very close to the true eigenvalues. Whether or not they are quantum ergodic would reflect the extent to which they are sufficiently random combinations of quasi-modes and the extent to which the collection of quasi-modes in $I_{M,m}^h$ is Liouville distributed.

REFERENCES

- [BL] N. Burq and G. Lebeau, Injections de Sobolev probabilistes et applications, arXiv:1111.7310.
- [CV] Y. Colin de Verdière, Quasi-modes sur les variétés Riemanniennes compactes, *Invent. Math.* 43 (1977), 15-52.
- [DSP] W. Duke and R. Schulze-Pillot, Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. *Invent. Math.* 99 (1990), no. 1, 49-57.
- [EH] P. Erdős and R. R. P. Hall, On the angular distribution of Gaussian integers with fixed norm. *Paul Erdős memorial collection. Discrete Math.* 200 (1999), no. 1-3, 87-94.
- [OV] G. Olshanski and A. Vershik, Ergodic unitarily invariant measures on the space of infinite Hermitian matrices. *Contemporary mathematical physics*, 137 - 175, Amer. Math. Soc. Transl. Ser. 2, 175, Amer. Math. Soc., Providence, RI, 1996.
- [G] F. Götze, Lattice point problems and values of quadratic forms. *Invent. Math.* 157 (2004), no. 1, 195-226.
- [P] C. Pommerenke, Über die Gleichverteilung von Gitterpunkten auf m-dimensionalen Ellipsoiden. *Acta Arith.* 5 1959 227-257.
- [Po] G. Popov, Invariant tori, effective stability and quasimodes with exponentially small errors I- Birkhoff normal forms, *Ann. Henri Poincaré* 1 (2000), 223-248.
- [S] S. Samuel, $U(N)$ integrals, $1/N$, and the De Wit-t Hooft anomalies. *J. Math. Phys.* 21 (1980), no. 12, 2695- 2703.
- [SZ] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* 200 (1999), no. 3, 661-683.
- [W] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank. *J. Math. Phys.* 19(5), 999-1001 (1978).
- [Z1] S. Zelditch, Quantum ergodicity on the sphere. *Comm. Math. Phys.* 146 (1992), no. 1, 61-71.
- [Z2] S. Zelditch, A random matrix model for quantum mixing. *Internat. Math. Res. Notices* 1996, no. 3, 115-137.
- [Z3] S. Zelditch, Large N limits of coadjoint orbits (in preparation).

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2370, USA
E-mail address: `zelditch@math.northwestern.edu`